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Abstract. For Markov chains in continuous time Keilson(1979) has shown that the relationship between the ergodic exit time T_E and the ergodic sojourn time T_V is identical to the relationship between the residual lifetime and the underlying lifetime at ergodicity in renewal theory. Keilson's result relies upon the memoryless property of exponential distributions, and it would not hold true, in general, for semi-Markov processes. The purpose of this paper is to introduce a new performance measure called the ergodic residual exit time T_W so as to prove that the relationship between T_W and T_V for semi-Markov processes is identical to the relationship between T_E and T_V for Markov chains in continuous time.
Keywords. Ergodic flow rate, ergodic exit time, ergodic sojourn time, ergodic residual exit time, semi-Markov processes.

1 Introduction

For many applications of Markov chains in continuous time, it is often important to introduce various system performance measures by decomposing the state space $\mathcal{N}$ into two sets:G consisting of good states and B containing only bad states. A simple example would be the first passage time of $N(t)$ from a good state $m \in G$ to any bad state in B denoted by T_{mB}. More sophisticated performance measures of this sort have been introduced by Keilson(1979), represented by the ergodic exit time T_E and the ergodic sojourn time T_V. The former is the time until the system reaches any bad state in B given that the system has been running since time immemorial and has been unobserved since its inception but is known to be in the good set G. The latter is similar except that it is known not only to be in the good set G but also just to have had a transition from a bad state in B to a good state in G. It is shown in Keilson(1979) that the relationship between T_E and T_V is identical to the relationship between the residual lifetime and the underlying lifetime at ergodicity in renewal theory.

Keilson's theorem concerning the relationship between T_E and T_V relies upon the memoryless property of exponential distributions. Accordingly, the theorem, in general, would not hold true for semi-Markov processes. The purpose of this paper is to introduce a new performance measure called the ergodic residual exit time T_W so as to prove that the relationship between T_W and T_V for semi-Markov processes is identical to the relationship between T_E and T_V for Markov chains in continuous time. Naturally, this would indicate that T_E is equal in distribution to T_W for Markov chains in continuous time. Indeed, a direct formal proof would be provided for this statement. The structure of this paper is as follows. In Section 2, a succinct summary of key related results for Markov chains in continuous time is extracted from Keilson(1979). The semi-Markov counterparts of those key results are then established in Section 3 with the ergodic residual exit time T_W newly introduced.

Throughout the paper, vectors and matrices are indicated by underbar and doubleunderbar respectively, e.g. $\underline{e}, \underline{q}, \underline{\underline{A}}_0, \underline{\underline{A}}(x)$, etc. Subvectors and submatrices are indexed by relevant sets, e.g. $\underline{e}_G = [e_m]_{m\in G}$, $\underline{\underline{A}}_{0:GB} = [A_{0:mn}]_{m\in G, n\in B}$, etc. The vector with all components equal to 1 is denoted by $\underline{1}$ and the zero vector by $\underline{0}$. The identity matrix is denoted by $\underline{\underline{I}}$. We also define $\delta_{mn} = 1$ if $m = n$, and $\delta_{mn} = 0$ otherwise.

2 Ergodic flow rate, ergodic exit time and ergodic sojourn time for Markov chains in continuous time

Let $\{N(t) : t \geq 0\}$ be an ergodic Markov chains in continuous time on $\mathcal{N} = \{0, 1, \ldots, N\}$ governed by a hazard rate matrix $\underline{\underline{\nu}} = [\nu_{mn}]$. Let G and B be two subsets of $\mathcal{N}$ satisfying $\mathcal{N} = G \cup B, G \neq \emptyset, B \neq \emptyset$ and $G \cap B = \emptyset$. With $\underline{\underline{\nu}}_D = [\delta_{mn}\nu_m]$ where $\nu_m = \sum_{n\in\mathcal{N}} \nu_{mn}$, the infinitesimal generator $\underline{\underline{Q}}$ of $N(t)$ and the ergodic probability vector $\underline{e}^T$ then satisfy

$$\underline{\underline{Q}} = -\underline{\underline{\nu}}_D + \underline{\underline{\nu}}\,; \qquad \underline{e}^T\underline{\underline{Q}} = \underline{0}^T\,. \tag{1}$$

In this section, we summarize key results of relevance to this paper concerning $N(t)$ from Keilson(1979).

Let T_{mB} be the first passage time of $N(t)$ from $m \in G$ to any state in B. Let $\sigma_{mB}(s) = \mathrm{E}[\mathrm{e}^{-sT_{mB}}]$ and define the vector $\underline{\sigma}_{G\to B}(s) = [\sigma_{mB}(s)]_{m\in G}$. One then has

$$\underline{\sigma}_{G\to B}(s) = \left[s\underline{\underline{I}}_{GG} - \underline{\underline{Q}}_{GG}\right]^{-1} \underline{\underline{\nu}}_{GB}\underline{1}_B\,. \tag{2}$$

The ergodic flow rate i_{mn} of $N(t)$ from state m to state n is the asymptotic renewal density for transitions from m to n, which is given by

$$i_{mn} = e_m\nu_{mn}\,. \tag{3}$$

Concerning the ergodic flow rate, the following set balance equation holds true.

$$i(G \to B) \stackrel{\text{def}}{=} \sum_{m\in G}\sum_{n\in B} i_{mn} = \sum_{m\in G}\sum_{n\in B} i_{nm} \stackrel{\text{def}}{=} i(G \leftarrow B)\,. \tag{4}$$

From (3), the set balance equation can be rewritten in matrix form as

$$\underline{e}_G^T \underline{\underline{\nu}}_{GB} \underline{1}_B = \underline{e}_B^T \underline{\underline{\nu}}_{BG} \underline{1}_G \, . \tag{5}$$

The ergodic exit time T_E of $N(t)$ from G is defined as the time until the system reaches any bad state in B given that the system has been running since time immemorial and has been unobserved since its inception but is known to be in the good set G. More formally, if we define $\sigma_E(s) = \mathrm{E}[\mathrm{e}^{-sT_E}]$, one has

$$\sigma_E(s) \stackrel{\text{def}}{=} \frac{\sum_{m \in G} e_m \sigma_{mB}(s)}{P(G)} = \frac{\underline{e}_G^T \underline{\sigma}_{G \to B}(s)}{P(G)} \, ; \qquad P(G) = \sum_{m \in G} e_m \, . \tag{6}$$

The ergodic sojourn time T_V is similar to T_E except that it is known not only to be in the good set G but also just to have had a transition from some bad state in B to some good state in G. By defining $i_{m \leftarrow B} = \sum_{n \in B} i_{nm}$ and $\underline{i}_{G \leftarrow B}^T = [i_{m \leftarrow B}]_{m \in G}$, the Laplace transform $\sigma_V(s) = \mathrm{E}[\mathrm{e}^{-sT_V}]$ is given formally as

$$\sigma_V(s) \stackrel{\text{def}}{=} \frac{\sum_{m \in G} i_{m \leftarrow B} \sigma_{mB}(s)}{i(G \leftarrow B)} = \frac{\underline{i}_{G \leftarrow B}^T \underline{\sigma}_{G \to B}(s)}{i(G \leftarrow B)} \, . \tag{7}$$

It is shown in Keilson(1979) that the relationship between T_E and T_V is identical to the relationship between the residual lifetime and the underlying lifetime at ergodicity in renewal theory. More specifically, with $\mu_V = \mathrm{E}[T_V]$, one has

$$\sigma_E(t) = \frac{1 - \sigma_V(t)}{s \cdot \mu_V} \, . \tag{8}$$

Keilson's proof of (8) hinges on the memoryless property of exponential variates. Accordingly, one cannot expect (8) to hold true for semi-Markov processes. In the next section, we newly introduce the ergodic residual exit time T_W for semi-Markov processes and prove that the relationship between T_W and T_V is identical to that between T_E and T_V for Markov chains in continuous time.

3 Ergodic flow rate, ergodic residual exit time and ergodic sojourn time for semi-Markov processes

The study of semi-Markov processes dates back to the middle of 50's, represented by the original papers by Levy(1954), Smith(1955) and Takacs(1954). Since then, various aspects of semi-Markov processes have been studied by many researchers. The reader is referred to two excellent survey papers by Çinlar(1969,1975). One of the important areas for the study of semi-Markov processes would be to see how certain properties of Markov chains in continuous time could be carried over to those of semi-Markov processes. In this section, we recapture the results for Markov chains in continuous time discussed in Section 2 within the context of semi-Markov processes. The first step for this purpose would be to derive ergodic flow rates for semi-Markov processes.

Let $\{J(t) : t \geq 0\}$ be an ergodic semi-Markov process on $\mathcal{N}$ characterized by a matrix p.d.f. $\underline{\underline{a}}(x) = [a_{mn}(x)]$ with its Laplace transform $\underline{\underline{\alpha}}(s) = \int_0^\infty e^{-sx}\underline{\underline{a}}(x)dx$. For notational convenience, we define $\underline{\underline{A}}_0 = \underline{\underline{\alpha}}(0)$ and $\underline{\underline{A}}_1 = -\frac{d}{ds}\underline{\underline{\alpha}}(s)|_{s=0}$. Let $\underline{q}^T$ be the ergodic probability vector of the discrete time Markov chain governed by $\underline{\underline{A}}_0$, i.e.

$$\underline{q}^T = \underline{q}^T \underline{\underline{A}}_0 \, ; \qquad \underline{q}^T > \underline{0} \, ; \qquad \underline{q}^T \underline{1} = 1 \, . \tag{9}$$

For $\mu_m = \sum_{n \in \mathcal{N}} A_{1:mn}$, we define $\underline{\underline{A}}_{D:1} = [\delta_{mn}\mu_m]$. Then the ergodic probability vector $\underline{e}^T$ of $J(t)$ can be expressed in terms of $\underline{q}^T$ as

$$\underline{e}^T = \frac{1}{M}\underline{q}^T \underline{\underline{A}}_{D:1} \, ; \qquad M = \underline{q}^T \underline{\underline{A}}_1 \underline{1} \, . \tag{10}$$

Let $N_{mn}(t)$ be the number of transitions of $J(t)$ from m to n in $[0, t]$. The ergodic flow rate i_{mn} of $J(t)$ is defined as

$$i_{mn} = \lim_{t \to \infty} \frac{\mathrm{E}[N_{mn}(t)]}{t} \, . \tag{11}$$

One then has the following theorem.

Theorem 1.

$$i_{mn} = \frac{1}{M} q_m A_{0:mn} \, .$$

Proof. Let $J^*(t)$ be a semi-Markov process on $\mathcal{N} \times \mathcal{N}$ governed by $\underline{\underline{b}}(x) = [b_{(m,\ell)(r,n)}(x)]$ where $b_{(m,\ell)(r,n)}(x) = \delta_{\ell r} a_{rn}(x)$. We note that $J^*(t)$ is constructed from $J(t)$ by coupling its two consecutive transitions. As in (9) and (10), for $J^*(t)$, one has $\underline{q}^{*T} = \underline{q}^{*T}\underline{\underline{B}}_0$ and $\underline{e}^{*T} = \frac{1}{M^*}\underline{q}^{*T}\underline{\underline{B}}_{D:1}$. It then follows that $M^* = \underline{q}^{*T}\underline{\underline{B}}_1\underline{1} = \underline{q}^T\underline{\underline{A}}_1\underline{1} = M$ and hence

$$e^*_{(m,n)} = \frac{1}{M} q_m A_{0:mn} \mu_n \, . \tag{12}$$

We now consider an alternating renewal process $\hat{J}(t)$ where $\hat{J}(t) = 0$ if $J^*(t) \neq (m, n)$ and $\hat{J}(t) = 1$ if $J^*(t) = (m, n)$. Let $D_{(m,n):i}$ be the dwell time of $\hat{J}(t)$ in state i for $i \in \{0, 1\}$. It should be noted that $\mathrm{E}[D_{(m,n):1}] = \mu_n$. From the classical theory of the alternating renewal process, one then sees from (11) that

$$i_{mn} = \frac{\mathrm{E}[D_{(m,n):1}]}{\mathrm{E}[D_{(m,n):0}] + \mathrm{E}[D_{(m,n):1}]} \cdot \frac{1}{\mathrm{E}[D_{(m,n):1}]} = e^*_{(m,n)} \frac{1}{\mu_n} \, . \tag{13}$$

The theorem now follows by substituting (12) into the last term in (13).

From Theorem 1, it can be readily seen that

$$\underline{i}^T_{G\leftarrow B} = [i_{m\leftarrow B}]^T_{m\in G} = \frac{1}{M}\underline{q}^T_B \underline{\underline{A}}_{0:BG}\,, \tag{14}$$

which in turn leads to the set balance equation for the semi-Markov ergodic flow rates as we show next.

Theorem 2. *For i_{mn} in Theorem 1, let $i(G \to B)$ and $i(G \leftarrow B)$ be defined as in (4). Then*

$$i(G \to B) = i(G \leftarrow B)\,.$$

Proof. From (9), one sees that $\underline{q}^T_G \underline{\underline{A}}_{0:GB}\underline{1}_B = \underline{q}^T_B \underline{\underline{A}}_{0:BG}\underline{1}_G$. It then follows from (14) and Theorem 1 that

$$i(G \leftarrow B) = \frac{1}{M}\underline{q}^T_B \underline{\underline{A}}_{0:BG}\underline{1}_G = \frac{1}{M}\underline{q}^T_G \underline{\underline{A}}_{0:GB}\underline{1}_B = i(G \to B)\,,$$

completing the proof.

The ergodic exit time and the ergodic sojourn time for semi-Markov processes can be defined as in (6) and (7), where $\underline{e}^T_G$ from (10) and $\underline{i}^T_{G\leftarrow B}$ from (14) should be employed, and $\sigma_{mB}(s)$ corresponds to the first passage time of the semi-Markov process from $m \in G$ to B. For the vector $\underline{\sigma}_{G\to B}(s) = [\sigma_{mB}]_{m\in G}$, one has, see e.g. Sumita and Masuda(1987),

$$\sigma_{G\to B}(s) = \left[\underline{\underline{I}}_{GG} - \underline{\underline{\alpha}}_{GG}(s)\right]^{-1} \underline{\underline{\alpha}}_{GB}(s)\underline{1}_B\,. \tag{15}$$

In order to observe the (residual lifetime)-vs-(lifetime) relationship for semi-Markov processes as in (8), we now introduce the ergodic residual exit time T_W of $J(t)$ from G defined as the time until the system reaches any bad state in B given that the system has been running since time immemorial and has been unobserved since its inception but is known to be in the good set G provided that the semi-Markov process has entered B at least once by now. More formally, if we define $\sigma_W(s) = \mathrm{E}[\mathrm{e}^{-sT_W}]$, one has

$$\sigma_W(s) \stackrel{\text{def}}{=} \frac{\sum_{m\in G} W_m(\frac{1-\sigma_{mB}(s)}{s\mu_{mB}})}{\sum_{m\in G} W_m}\,; \qquad W_m = \frac{i_{m\leftarrow B}\cdot \mu_{mB}}{i(G\leftarrow B)}\,. \tag{16}$$

We are now in a position to prove the following theorem.

Theorem 3.

$$\sigma_W(s) = \frac{1-\sigma_V(s)}{s\cdot \mu_V}\,.$$

Proof. From (7) and (16), one sees that

$$\mu_V = \frac{\sum_{m\in G} i_{m\leftarrow B}\mu_{mB}}{i(G\leftarrow B)} = \sum_{m\in G} W_m\,.$$

It then follows from (16) that

$$\sigma_W(s) = \frac{\sum_{m\in G} \frac{i_{m\leftarrow B}}{i(G\leftarrow B)}(\frac{1-\sigma_{mB}(s)}{s})}{\mu_V} = \frac{\sum_{m\in G} \frac{i_{m\leftarrow B}}{i(G\leftarrow B)}(1-\sigma_{mB}(s))}{s\cdot\mu_V}$$
$$= \frac{\sum_{m\in G} \frac{i_{m\leftarrow B}}{i(G\leftarrow B)} - \sum_{m\in G} \frac{i_{m\leftarrow B}\sigma_{mB}(s)}{i(G\leftarrow B)}}{s\cdot\mu_V} = \frac{1-\sigma_V(s)}{s\cdot\mu_V}\,,$$

proving the theorem.

Since a Markov chain in continuous time is a special case of a semi-Markov process, both (8) and Theorem 3 should hold true for such Markov chains. One then expects that $T_E \overset{\mathrm{d}}{=} T_W$, which we prove directly below.

Theorem 4. *For Markov chains in continuous time, one has*

$$T_E \overset{d}{=} T_W\,.$$

Proof. By differentiating Equation (15) with respect to s and then setting $s = 0$, one finds after a little algebra that

$$\underline{\mu}_{G\to B} = -\frac{d}{ds}\underline{\sigma}_{G\to B}(s)|_{s=0} = \left[\underline{\underline{I}}_{GG} - \underline{\underline{A}}_{0:GG}\right]^{-1} \underline{\underline{A}}_{D:1:GG}\underline{1}_G\,.$$

From (10), this result together with $\underline{q}_B^T\underline{\underline{A}}_{0:BG} = \underline{q}_G^T\left[\underline{\underline{I}}_{GG} - \underline{\underline{A}}_{0:GG}\right]$ then leads to

$$\sum_{m\in G} i_{Bm}\cdot\mu_{mB} = \frac{1}{M}\underline{q}_G^T\underline{\underline{A}}_{D:1:GG}\underline{1}_G = \sum_{r\in G} e_r = P(G)\,.$$

From (2) and (5), one finds that

$$\frac{1}{s}\underline{i}^T_{G\leftarrow B}[\underline{1}_G - \underline{\sigma}_{G\to B}(s)] = \underline{e}^T_G\underline{\sigma}_{G\to B}(s)\,.$$

The theorem now follows from (6) and (16).

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